

# Negativity and strong monogamy of multi-party quantum entanglement beyond qubits

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We propose the square of convex-roof extended negativity(SCREN) as a powerful candidate to characterize strong monogamy of multi-party quantum entanglement. We first provide a strong monogamy inequality of multi-party entanglement using SCREN, and show that the tangle-based multi-qubit strong monogamy inequality can be rephrased by SCREN. We further show that SCREN strong monogamy inequality is still true for the counterexamples that violate tangle-based strong monogamy inequality in higher-dimensional quantum systems rather than qubits. We also analytically show that SCREN strong monogamy inequality is true for a large class of multi-qudit states, a superposition of multi-qudit generalized W-class states and vacuums. Thus SCREN is a good alternative to characterize the strong monogamy of entanglement even in multi-qudit systems.

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## I. INTRODUCTION

Quantum entanglement is a quantum correlation used as a resource in various applications of quantum information theory such as quantum teleportation and quantum cryptography [1–3]. One important property of entanglement is its restricted shareability in multi-party quantum systems, which does not have any classical counterpart. This restriction of entanglement shareability among multi-party systems is known as the *monogamy of entanglement* (MoE) [4–8].

The first mathematical characterization of MoE was established by Coffman-Kundu-Wootters(CKW) as an inequality [4]; for a three-qubit pure state  $|\psi\rangle_{ABC}$ ,

$$\tau(|\psi\rangle_{ABC}) \geq \tau(\rho_{A|B}) + \tau(\rho_{A|C}),$$

where  $\tau(|\psi\rangle_{ABC})$  is the *one-tangle* of  $|\psi\rangle_{ABC}$  quantifying the pure state entanglement between  $A$  and  $BC$ , and  $\tau(\rho_{A|B})$  (similarly with  $\tau(\rho_{A|C})$ ) is the *two-tangle* of the reduced density matrix  $\rho_{AB} = \text{tr}_C|\psi\rangle_{ABC}\langle\psi|$  quantifying the two-qubit entanglement inherent in  $\rho_{AB}$ .

This inequality is also referred to as *CKW inequality*, and it shows the mutually exclusive nature of two-qubit entanglement shared in three-qubit systems; more entanglement shared between two qubits  $A$  and  $B$  leads to less entanglement between the other two qubits  $A$  and  $C$  so that their summation does not exceed the total entanglement between  $A$  and  $BC$ . Moreover, the residual entanglement from the difference between left and right-hand sides of CKW inequality is interpreted as the genuine three-qubit entanglement, *three-tangle*.

Later, CKW inequality was generalized for multi-qubit systems [5] as well as some cases of higher-dimensional

quantum systems [9–12]. A general monogamy inequality for arbitrary quantum systems was established in terms of the *squashed entanglement* [13, 14].

Recently, the definition of three-tangle was generalized into arbitrary  $n$ -qubit systems, namely  *$n$ -tangle* quantifying the genuine multi-qubit entanglement. By conjecturing the nonnegativity of the  $n$ -tangle, the concept of *strong monogamy*(SM) inequality of  $n$ -qubit entanglement was proposed [15]. Although an analytical proof of SM conjecture for arbitrary multi-qubit states seems to be a formidable challenge due to the numerous optimization processes arising in the definition of  $n$ -tangle, an extensive numerical evidence was presented for four qubit systems together with an analytical proof for some cases of multi-qubit systems [15, 16].

However, tangle is known to fail in the generalization of CKW inequality for higher dimensional quantum systems rather than qubits; there exist quantum states in  $3 \otimes 3 \otimes 3$  and even in  $3 \otimes 2 \otimes 2$  quantum systems violating CKW inequality [17, 18]. Because SM inequality proposed in [15] is reduced to CKW inequality for  $n = 3$ , these counterexamples of CKW inequality also implies the violation of SM inequality using tangles in higher-dimensional systems rather than qubits.

Here we propose the *square of convex-roof extended negativity*(SCREN) as a powerful candidate to characterize the strongly monogamous property of multi-qudit systems. We first provide a SM inequality of multi-party entanglement using SCREN, and show that the SM inequality of multi-qubit entanglement using tangle [15, 16] can be rephrased by SCREN. This SCREN SM inequality is also true for the counterexamples of tangle in higher-dimensional systems. Moreover, we analytically show that SCREN SM inequality is saturated by a large class of multi-qudit states, a superposition of multi-qudit generalized W-class states and vacuums. Thus SCREN is a good alternative for strong monogamy of multi-party entanglement even in higher-dimensional systems.

The paper is organized as follows. In Sec. II A, we re-

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view the definition of negativity, and provide the relation between tangle and SCREN for multi-qubit monogamy inequality in Sec. II B. In Sec. III A, we recall the multi-qubit SM inequality in terms of tangle, and propose a multi-qudit SM inequality using SCREN in III B. In Sec. IV A, we provide the definition of multi-qudit generalized W-class states as well as some useful properties of this class of states. In Sec. IV B, we analytically show that the SCREN SM inequality of multi-qudit entanglement is saturated by a superposition of generalized W-class states and vacuum. In Sec. V, we summarize our results.

## II. NEGATIVITY AND MONOGAMY OF MULTI-PARTY QUANTUM ENTANGLEMENT

### A. Negativity

For a bipartite pure state  $|\phi\rangle_{AB}$  in a  $d \otimes d'$  ( $d \leq d'$ ) quantum system with its Schmidt decomposition,

$$|\phi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |ii\rangle, \quad \lambda_i \geq 0, \quad \sum_{i=0}^{d-1} \lambda_i = 1, \quad (1)$$

its *negativity* is defined as

$$\mathcal{N}(|\phi\rangle_{AB}) = \left\| |\phi\rangle_{AB} \langle \phi|^{T_B} \right\|_1 - 1 = 2 \sum_{i < j} \sqrt{\lambda_i \lambda_j}, \quad (2)$$

where

$$|\phi\rangle_{AB} \langle \phi|^{T_B} = \sum_{i,j=0}^{d-1} \sqrt{\lambda_i \lambda_j} |ij\rangle_{AB} \langle ji| \quad (3)$$

is the partial transposition of  $|\phi\rangle_{AB}$  and  $\|\cdot\|_1$  is the trace norm [19].

Because the possible negative eigenvalues of the partially transposed state in Eq. (3) are  $-\sqrt{\lambda_i \lambda_j}$  for  $i < j$  with corresponding eigenstates  $|\psi_{ij}\rangle_{AB} = \frac{1}{\sqrt{2}}(|ij\rangle_{AB} - |ji\rangle_{AB})$ , the definition of negativity in Eq. (2) is thus the sum of all possible negative eigenvalues with a constant proportion [20]. Eq. (2) can also have an alternative definition as

$$\mathcal{N}(|\phi\rangle_{AB}) = 2 \sum_{i < j} \sqrt{\lambda_i \lambda_j} = (\text{tr} \sqrt{\rho_A})^2 - 1, \quad (4)$$

where  $\rho_A = \text{tr}_B |\phi\rangle_{AB} \langle \phi|$  is the reduced density matrix of  $|\phi\rangle_{AB}$  on subsystem A. For a bipartite mixed state  $\rho_{AB}$ , its negativity is analogously defined as

$$\mathcal{N}(\rho_{AB}) = \left\| \rho_{AB}^{T_B} \right\|_1 - 1, \quad (5)$$

where  $\rho_{AB}^{T_B}$  is the partial transposition of  $\rho_{AB}$ .

*Positive partial transposition* (PPT) [21, 22] gives a separability criterion for bipartite pure states and two-qubit mixed states. PPT is also a necessary and sufficient

condition for nondistillability in  $2 \otimes n$  quantum system [23, 24]. However, there also exist entangled mixed states with PPT in higher-dimensional quantum systems rather than  $2 \otimes 2$  or  $2 \otimes 3$  quantum systems. [23, 25]. For this case, negativity in Eq. (5) cannot distinguish PPT bound entangled states from separable states, and thus, negativity itself is not sufficient to be a good measure of entanglement even in a  $2 \otimes n$  quantum system.

One way to overcome this lack of separability criterion of negativity in higher-dimensional mixed quantum states is using *convex-roof extension* [26]; for a bipartite mixed state  $\rho_{AB}$ , its convex-roof extended negativity is

$$\mathcal{N}_m(\rho_{AB}) = \min_{\{p_k, |\phi_k\rangle\}} \sum_k p_k \mathcal{N}(|\phi_k\rangle_{AB}), \quad (6)$$

where the minimum is taken over all possible pure state decompositions of  $\rho_{AB} = \sum_k p_k |\phi_k\rangle_{AB} \langle \phi_k|$ . Convex-roof extended negativity gives a perfect discrimination of PPT bound entangled states and separable states in any bipartite quantum system. Moreover, it was also shown that the quantity in Eq. (6) cannot be increased by local quantum operations and classical communications (LOCC) [9, 26].

### B. Monogamy Inequality Using Negativity

For a two-qubit pure state  $|\psi\rangle_{AB}$  [27], its tangle (or one-tangle) is defined as

$$\tau(|\psi\rangle_{AB}) = 4 \det \rho_A, \quad (7)$$

with the reduced density matrix  $\rho_A = \text{tr}_B |\psi\rangle_{AB} \langle \psi|$ . For a two-qubit mixed state  $\rho_{AB}$ , its tangle (or two-tangle) is defined as

$$\tau(\rho_{AB}) = \left[ \min_{\{p_h, |\psi_h\rangle\}} \sum_h p_h \sqrt{\tau(|\psi_h\rangle_{AB})} \right]^2 \quad (8)$$

where the minimization is taken over all possible pure state decompositions

$$\rho_{AB} = \sum_h p_h |\psi_h\rangle_{AB} \langle \psi_h|. \quad (9)$$

Mathematically, monogamy of multi-party quantum entanglement was first characterized in three-qubit systems by Coffman, Kundu and Wootters (CKW) [4]; using one and two tangles as the bipartite entanglement quantification, monogamy inequality of three-qubit entanglement was proposed as

$$\tau(|\psi\rangle_{ABC}) \geq \tau(\rho_{AB}) + \tau(\rho_{AC}), \quad (10)$$

where  $\tau(|\psi\rangle_{ABC})$  is the one tangle of the three-qubit pure state  $|\psi\rangle_{ABC}$  quantifying the bipartite entanglement between A and BC, and  $\tau(\rho_{AB})$  and  $\tau(\rho_{AC})$  are

the two tangles of the two-qubit reduced states  $\rho_{AB} = \text{tr}_C |\psi\rangle_{ABC} \langle \psi|$  and  $\rho_{AC} = \text{tr}_B |\psi\rangle_{ABC} \langle \psi|$ , respectively.

Later CKW inequality in (10) was generalized into  $n$ -qubit systems [5] as

$$\tau \left( |\psi\rangle_{A_1|A_2\cdots A_n} \right) \geq \sum_{j=2}^n \tau \left( \rho_{A_1|A_j} \right), \quad (11)$$

for one tangle  $\tau \left( |\psi\rangle_{A_1|A_2\cdots A_n} \right)$  and two tangles  $\tau \left( \rho_{A_1|A_j} \right)$  of each two-qubit reduced density matrices  $\rho_{A_1 A_j}$  on subsystems  $A_1 A_j$  for each  $j = 2, \dots, n$ . However, tangle is known to fail in the generalization of CKW inequality for higher dimensional quantum systems rather than qubits; there exist quantum states in  $3 \otimes 3 \otimes 3$  and even in  $3 \otimes 2 \otimes 2$  quantum systems violating CKW inequality in (10) [17, 18].

Now we consider another generalization of tangles from qubits to qudit systems using negativity [9]. We first note that for any pure state  $|\psi\rangle_{AB}$  with Schmidt-rank 2 (especially for two-qubit pure state)

$$|\psi\rangle_{AB} = \sqrt{\lambda_1} |e_0\rangle_A \otimes |f_0\rangle_B + \sqrt{\lambda_2} |e_1\rangle_A \otimes |f_1\rangle_B, \quad (12)$$

the *square* of negativity in Eq. (4) coincides with the tangle in Eq. (7)

$$\mathcal{N}^2 \left( |\psi\rangle_{A|B} \right) = 4\lambda_1 \lambda_2 = \tau \left( |\psi\rangle_{A|B} \right). \quad (13)$$

Thus the two-tangle of any two-qubit state  $\rho_{AB}$  in Eq. (8) can be rephrased as

$$\tau \left( \rho_{A|B} \right) = \left[ \min_{\{p_h, |\psi_h\rangle\}} \sum_h p_h \mathcal{N} \left( |\psi_h\rangle_{A|B} \right) \right]^2 \quad (14)$$

where the right-hand side of Eq. (14) is square of the convex-roof extended negativity in Eq. (6). Based on this idea, we propose a bipartite entanglement measure using negativity; for any two-qudit mixed state  $\rho_{AB}$  its *square of convex-roof extended negativity* (SCREN) is defined as

$$\mathcal{N}_{sc}(\rho_{A|B}) = \left[ \min_{\{p_h, |\psi_h\rangle\}} \sum_h p_h \mathcal{N} \left( |\psi_h\rangle_{A|B} \right) \right]^2. \quad (15)$$

From the properties of convex-roof extended negativity in Eq. (6), it is straightforward to check that SCREN has the separability criterion and monotonicity under LOCC [26]. We also note that Eqs. (14) and (15) imply the coincidence of SCREN with two-tangle for any two-qubit state  $\rho_{AB}$ ,

$$\mathcal{N}_{sc}(\rho_{A|B}) = \tau \left( \rho_{A|B} \right) \quad (16)$$

Consequently, the multi-qubit monogamy inequality in terms of tangles in (11) can be rephrased in terms of SCREN as,

$$\mathcal{N}_{sc} \left( |\psi\rangle_{A_1|A_2\cdots A_n} \right) \geq \sum_{j=2}^n \mathcal{N}_{sc} \left( \rho_{A_1|A_j} \right). \quad (17)$$

Moreover, Inequality (17) still holds for the counterexamples [17, 18] that violate CKW inequality in higher-dimensional systems [9]. Thus SCREN is a good generalization of two-qubit tangle into higher-dimensional quantum systems without any known counterexamples even in higher-dimensional quantum systems so far.

### III. STRONG MONOGAMY OF MULTI-PARTY QUANTUM ENTANGLEMENT

#### A. Multi-Qubit Strong Monogamy Inequality

For any three-qubit pure state  $|\psi\rangle_{ABC}$ , the residual entanglement from the difference between left and right-hand sides of CKW Inequality (10) is also interpreted as the genuine three-party entanglement, namely *three-tangle* of  $|\psi\rangle_{ABC}$

$$\tau \left( |\psi\rangle_{A|B|C} \right) = \tau \left( |\psi\rangle_{A|BC} \right) - \tau \left( \rho_{A|B} \right) - \tau \left( \rho_{A|C} \right). \quad (18)$$

The three-tangle in Eq. (18) is a good measure of genuine three-qubit entanglement, which is invariant under the permutation of subsystems  $A$ ,  $B$  and  $C$  [28].

The definition of three-tangle was generalized for arbitrary  $n$ -qubit quantum states [15]; for an  $n$ -qubit pure state  $|\psi\rangle_{A_1 A_2 \cdots A_n}$ , its *n-tangle* is defined as

$$\tau \left( |\psi\rangle_{A_1|A_2|\cdots|A_n} \right) = \tau \left( |\psi\rangle_{A_1|A_2\cdots A_n} \right) - \sum_{m=2}^{n-1} \sum_{\vec{j}_m} \tau \left( \rho_{A_1|A_{j_1^m}|\cdots|A_{j_{m-1}^m}} \right)^{m/2}, \quad (19)$$

where the index vector  $\vec{j}^m = (j_1^m, \dots, j_{m-1}^m)$  spans all the ordered subsets of the index set  $\{2, \dots, n\}$  with  $(m-1)$  distinct elements. Eq. (19) is a recurrent definition that needs all the  $m$  tangles  $\tau \left( \rho_{A_1|A_{j_1^m}|\cdots|A_{j_{m-1}^m}} \right)$  of  $m$ -qubit reduced density matrices  $\rho_{A_1 A_{j_1^m} \cdots A_{j_{m-1}^m}}$  for  $2 \leq m \leq n-1$ , where  $\tau \left( \rho_{A_1|A_{j_1^m}|\cdots|A_{j_{m-1}^m}} \right)$  is defined as

$$\tau\left(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}}\right) = \left[ \min_{\{p_h, |\psi_h\rangle\}} \sum_h p_h \sqrt{\tau\left(|\psi_h\rangle_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}}\right)} \right]^2, \quad (20)$$

with the minimization over all possible pure state decompositions

$$\rho_{A_1 A_{j_1^m} \dots A_{j_{m-1}^m}} = \sum_h p_h |\psi_h\rangle_{A_1 A_{j_1^m} \dots A_{j_{m-1}^m}} \langle \psi_h|. \quad (21)$$

For  $n = 3$ , the definition of  $n$ -tangle in Eq. (19) reduces to that of three-tangle in Eq. (18) whose nonnegativity is equivalent to the CKW inequality (10). In other words, the nonnegativity of three-tangle provides us with a quantitative characterization of three-qubit monogamy of entanglement. For  $n = 2$ , Eq. (20) also reduces to the two-tangle of two-qubit state  $\rho_{A_1 A_2}$  in Eq. (8).

Based on this idea, a *strong monogamy* (SM) inequality of multi-qubit entanglement was proposed as

$$\tau\left(|\psi\rangle_{A_1|A_2\dots A_n}\right) \geq \sum_{m=2}^{n-1} \sum_{\vec{j}_m} \tau\left(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}}\right)^{m/2} \quad (22)$$

by conjecturing the nonnegativity of  $n$ -tangle in Eq. (19). The lower term in Inequality (22) appears in between the both sides of the  $n$ -qubit CKW inequality in (11) as

$$\begin{aligned} \tau\left(|\psi\rangle_{A_1|A_2\dots A_n}\right) &\geq \sum_{j=2}^n \tau\left(\rho_{A_1|A_j}\right) \\ &\quad + \sum_{m=3}^{n-1} \sum_{\vec{j}_m} \tau\left(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}}\right)^{m/2} \\ &\geq \sum_{j=2}^n \tau\left(\rho_{A_1|A_j}\right), \end{aligned} \quad (23)$$

therefore it is a *stronger* inequality. We also note that Inequality (22) encapsulates three-qubit CKW inequality in (10) for  $n = 3$ . Thus Inequality (22) is another generalization of three-qubit monogamy inequality into multi-qubit systems in a stronger form.

For the validity of SM inequality in (22), an extensive numerical evidence was presented for four qubit systems together with analytical proof for some cases of multi-qubit systems. It was also recently shown that Inequality (22) is also true for a large class of multi-qubit generalized W-class states,

$$|\psi\rangle_{A_1 A_2 \dots A_n} = a_1 |10 \dots 0\rangle + a_2 |01 \dots 0\rangle + \dots + a_n |00 \dots 1\rangle \quad (24)$$

with  $\sum_{i=1}^n |a_i|^2 = 1$  [16].

## B. SCREN Strong Monogamy Inequality

Although Inequality (22) proposes a stronger monogamous property of multi-qubit entanglement with various cases of analytic proof, Inequality (22) is no longer valid for higher-dimensional quantum systems rather than qubits; for  $n = 3$ , Inequality (22) becomes a CKW-type inequality of three-party quantum systems,

$$\tau\left(|\psi\rangle_{A|BC}\right) \geq \tau\left(\rho_{A|B}\right) + \tau\left(\rho_{A|C}\right). \quad (25)$$

However, it is also known that there exists a pure state in  $3 \otimes 2 \otimes 2$  quantum systems [9, 18],

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{6}}(\sqrt{2}|010\rangle + \sqrt{2}|101\rangle + |200\rangle + |211\rangle), \quad (26)$$

where  $\tau\left(|\psi\rangle_{A|BC}\right) = \frac{12}{9}$  and  $\tau\left(\rho_{A|B}\right) = \tau\left(\rho_{A|C}\right) = \frac{8}{9}$ , therefore

$$\tau\left(|\psi\rangle_{A|BC}\right) < \tau\left(\rho_{A|B}\right) + \tau\left(\rho_{A|C}\right). \quad (27)$$

In other words, the counterexample for three-party CKW inequality in Eq. (26) is also a counterexample for SM inequality in (22) in higher-dimensional quantum systems rather than qubits. Thus tangle-based SM inequality can only be valid for multi-qubit systems and even a tiny extension in any of the subsystems leads to a violation.

Here we propose another generalization of multi-qubit SM inequality into higher-dimensional quantum systems using SCREN. Due to the coincidence of tangle and SCREN for two-qubit states and any pure state of Schmidt-rank two in Eq. (16), the definition of three-tangle in Eq. (18) can be naturally rephrased in terms of SCREN; for any three-qubit pure state  $|\psi\rangle_{ABC}$ ,

$$\begin{aligned} \mathcal{N}_{sc}\left(|\psi\rangle_{A|B|C}\right) &= \mathcal{N}_{sc}\left(|\psi\rangle_{A|BC}\right) \\ &\quad - \mathcal{N}_{sc}\left(\rho_{A|B}\right) - \mathcal{N}_{sc}\left(\rho_{A|C}\right). \end{aligned} \quad (28)$$

For analogous terminology, we denote  $\mathcal{N}_{sc}\left(|\psi\rangle_{A|B|C}\right)$  in Eq. (28) as *three-SCREN* where  $\mathcal{N}_{sc}\left(|\psi\rangle_{A|BC}\right)$  and  $\mathcal{N}_{sc}\left(\rho_{A|B}\right)$  are *one-* and *two-SCREN*, respectively.

Now we generalize the definition of three-SCREN in Eq. (28) into arbitrary multi-party, higher-dimensional quantum systems. For an  $n$ -qudit pure state  $|\psi\rangle_{A_1 A_2 \dots A_n}$ , its  $n$ -SCREN is defined as

$$\mathcal{N}_{sc}(|\psi\rangle_{A_1|A_2|\dots|A_n}) = \mathcal{N}_{sc}(|\psi\rangle_{A_1|A_2|\dots|A_n}) - \sum_{m=2}^{n-1} \sum_{\vec{j}_m} \mathcal{N}_{sc}(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}})^{m/2}, \quad (29)$$

where  $\mathcal{N}_{sc}(|\psi\rangle_{A_1|A_2|\dots|A_n})$  is the one-SCREN of  $n$ -qudit pure state with respect to the bipartition between  $A_1$  and

the other qudit systems, and the  $m$ -SCREN of  $m$ -qubit reduced density matrix  $\rho_{A_1 A_{j_1^m} \dots A_{j_{m-1}^m}}$  is defined as

$$\mathcal{N}_{sc}(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}}) = \left[ \min_{\{p_h, |\psi_h\rangle\}} \sum_h p_h \sqrt{\mathcal{N}_{sc}(|\psi_h\rangle_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}})} \right]^2, \quad (30)$$

with the minimization over all possible pure state decompositions of  $\rho_{A_1 A_{j_1^m} \dots A_{j_{m-1}^m}}$ . We also note that the index vector  $\vec{j}_m = (j_1^m, \dots, j_{m-1}^m)$  in the second summation of Eq. (29) spans all the ordered subsets of the index set  $\{2, \dots, n\}$  with  $(m-1)$  distinct elements.

For a multi-qudit pure state  $|\psi\rangle_{A_1 A_2 \dots A_n}$ , the *SCREN-SM inequality* of multi-party entanglement can be derived as

$$\mathcal{N}_{sc}(|\psi\rangle_{A_1|A_2|\dots|A_n}) \geq \sum_{m=2}^{n-1} \sum_{\vec{j}_m} \mathcal{N}_{sc}(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}})^{m/2}, \quad (31)$$

conjecturing the nonnegativity of  $n$ -SCREN in Eq. (29). From the relation of SCREN and tangle in Eq. (16), Inequality (31) is reduced to Inequality (22) for any multi-qubit states. Thus Inequality (31) is a generalization of multi-qubit SM inequality in terms of tangle, which is valid for the classes of multi-qubit quantum states considered in [15, 16].

For the counterexample of CKW inequality in Eq. (26), it is straightforward to check  $\mathcal{N}_{sc}(|\psi\rangle_{A|BC}) = 4$  whereas  $\mathcal{N}_{sc}(\rho_{A|B}) = \mathcal{N}_{sc}(\rho_{A|C}) = \frac{8}{9}$ , and thus

$$\mathcal{N}_{sc}(|\psi\rangle_{A|BC}) \geq \mathcal{N}_{sc}(\rho_{A|B}) + \mathcal{N}_{sc}(\rho_{A|C}). \quad (32)$$

Moreover, for the other counterexample in  $3 \otimes 3 \otimes 3$  quantum systems [17],

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle), \quad (33)$$

we have  $\mathcal{N}_{sc}(|\psi\rangle_{A|BC}) = 4$  whereas  $\mathcal{N}_{sc}(\rho_{A|B}) = \mathcal{N}_{sc}(\rho_{A|C}) = 1$ . In other words, Inequality (32) is still true for all the known counterexamples of CKW inequality, therefore SCREN is a good alternative of tangle in characterizing strongly monogamous property of multi-party entanglement.

#### IV. SCREN STRONG MONOGAMY INEQUALITY OF MULTI-QUDIT ENTANGLEMENT

##### A. Multi-Qudit Generalized W-class States

Let us recall the definition of multi-qudit generalized W-class state [18],

$$|W_n^d\rangle_{A_1 \dots A_n} = \sum_{i=1}^{d-1} (a_{1i}|i0 \dots 0\rangle + a_{2i}|0i \dots 0\rangle + \dots + a_{ni}|00 \dots 0i\rangle), \quad (34)$$

with the normalization condition  $\sum_{s=1}^n \sum_{i=1}^{d-1} |a_{si}|^2 = 1$ . The state in Eq. (34) is a coherent superposition of all  $n$ -qudit product states with Hamming weight one. We also note that the term “generalized” naturally arises because Eq. (34) includes  $n$ -qubit W-class states in Eq. (24) as a special case when  $d = 2$ .

Before we further investigate strongly monogamous property of entanglement for this generalized W-class state, we first recall a very useful property of quantum states proposed by Hughston-Jozsa-Wootters (HJW) showing the unitary freedom in the ensemble for density matrices [29].

**Proposition 1.** (*HJW theorem*) *The sets  $\{|\tilde{\phi}_i\rangle\}$  and  $\{|\tilde{\psi}_j\rangle\}$  of (possibly unnormalized) states generate the same density matrix if and only if*

$$|\tilde{\phi}_i\rangle = \sum_j u_{ij} |\tilde{\psi}_j\rangle \quad (35)$$

where  $(u_{ij})$  is a unitary matrix of complex numbers, with indices  $i$  and  $j$ , and we pad whichever set of states  $\{|\tilde{\phi}_i\rangle\}$  or  $\{|\tilde{\psi}_j\rangle\}$  is smaller with additional zero vectors so that the two sets have the same number of elements.

A direct consequence of Proposition 1 is the following; for two pure-state decompositions  $\sum_i p_i |\phi_i\rangle\langle\phi_i|$  and

$\sum_j q_j |\psi_j\rangle\langle\psi_j|$ , they represent the same density matrix, that is  $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i| = \sum_j q_j |\psi_j\rangle\langle\psi_j|$  if and only if  $\sqrt{p_i} |\phi_i\rangle = \sum_j u_{ij} \sqrt{q_j} |\psi_j\rangle$  for some unitary matrix  $u_{ij}$ . Using Proposition 1, we provide the following lemma, which shows a structural property of multi-qudit generalized W-class states.

**Lemma 1.** *Let  $|\psi\rangle_{A_1 \dots A_n}$  be a  $n$ -qudit pure state in a superposition of a  $n$ -qudit generalized W-class state in Eq. (24) and vacuum, that is,*

$$|\psi\rangle_{A_1 A_2 \dots A_n} = \sqrt{p} |W_n^d\rangle_{A_1 \dots A_n} + \sqrt{1-p} |0 \dots 0\rangle_{A_1 \dots A_n} \quad (36)$$

for  $0 \leq p \leq 1$ . Let  $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  be a reduced density matrix of  $|\psi\rangle_{A_1 \dots A_n}$  onto  $m$ -qudit subsystems  $A_1 A_{j_1} \dots A_{j_{m-1}}$  with  $2 \leq m \leq n-1$ . For any pure state decomposition of  $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  such that

$$\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}} = \sum_k q_k |\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle\phi_k|, \quad (37)$$

$|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  is a superposition of a  $m$ -qudit generalized W-class state and vacuum.

*Proof.* Due to the symmetry of the structure of multi-qudit generalized W-class states with respect to permuting subsystems, here we only consider the reduced density matrix  $\rho_{A_1 A_2 \dots A_m}$  of the first  $m$  qudits subsystems  $A_1 A_2 \dots A_m$ , where the general cases of  $m$ -qudit subsystems  $A_1 A_{j_1} \dots A_{j_{m-1}}$  is then analogously following.

From a straightforward calculation, we obtain

$$\rho_{A_1 A_2 \dots A_m} = |\tilde{x}\rangle_{A_1 A_2 \dots A_m} \langle\tilde{x}| + |\tilde{y}\rangle_{A_1 A_2 \dots A_m} \langle\tilde{y}|, \quad (38)$$

where

$$\begin{aligned} |\tilde{x}\rangle_{A_1 A_2 \dots A_m} &= \sqrt{p} \sum_{i=1}^{d-1} (a_{1i} |i0 \dots 0\rangle_{A_1 A_2 \dots A_m} + a_{2i} |0i0 \dots 0\rangle_{A_1 A_2 \dots A_m} + \dots + a_{mi} |00 \dots i\rangle_{A_1 A_2 \dots A_m}) \\ &\quad + \sqrt{1-p} |00 \dots 0\rangle_{A_1 A_2 \dots A_m}, \\ |\tilde{y}\rangle_{A_1 A_2 \dots A_m} &= \sqrt{p \sum_{i=1}^{d-1} (|a_{m+1i}|^2 + \dots + |a_{ni}|^2)} |00 \dots 0\rangle_{A_1 A_2 \dots A_m} \end{aligned} \quad (39)$$

are the unnormalized states in  $m$ -qubit subsystems  $A_1 A_2 \dots A_m$ .

Now, let us consider the unnormalized states  $|\tilde{\phi}_k\rangle_{A_1 A_2 \dots A_m} = \sqrt{q_k} |\phi_k\rangle_{A_1 A_2 \dots A_m}$  for each  $k$  in the pure-state decomposition Eq. (37). From Proposition 1, there exists an  $r \times r$  unitary matrix  $(u_{kl})$  such that

$$|\tilde{\phi}_k\rangle_{A_1 A_2 \dots A_m} = u_{k1} |\tilde{x}\rangle_{A_1 A_2 \dots A_m} + u_{k2} |\tilde{y}\rangle_{A_1 A_2 \dots A_m}, \quad (40)$$

for each  $k$ . Moreover, Eqs. (39) imply that both  $|\tilde{x}\rangle_{A_1 A_2 \dots A_m}$  and  $|\tilde{y}\rangle_{A_1 A_2 \dots A_m}$  are linear combinations of  $m$ -qudit generalized W-class states and vacuums. In other words,  $|\tilde{\phi}_k\rangle_{A_1 A_2 \dots A_m}$  in Eq. (40) is an unnormalized superposition of a  $m$ -qudit generalized W-class state and vacuum for each  $k$ . Thus the same is true for the normalized state  $|\phi_k\rangle_{A_1 A_2 \dots A_m}$  for each  $k$ .  $\square$

## B. SCREN Strong Monogamy Inequality and Generalized W-class States

In this section, we prove that the multi-qudit SCREN SM inequality of entanglement is true for a large class

of multi-qubit quantum states in Eq. (36); superposition of multi-qudit generalized W-class states and vacuums. We first provide the following theorem about the multi-qudit generalized W-class and the CKW-type monogamy inequality.

**Theorem 2.** *For a  $n$ -qudit pure state*

$$|\psi\rangle_{A_1 A_2 \dots A_n} = \sqrt{p} |W_n^d\rangle_{A_1 \dots A_n} + \sqrt{1-p} |0 \dots 0\rangle_{A_1 \dots A_n} \quad (41)$$

where  $|W_n^d\rangle_{A_1 \dots A_n}$  is a  $n$ -qudit generalized W-class state in Eq. (24) and  $|0 \dots 0\rangle_{A_1 \dots A_n}$  is the vacuum, we have

$$\mathcal{N}_{sc}(|\psi\rangle_{A_1 | A_2 \dots A_n}) = \mathcal{N}_{sc}(\rho_{A_1 | A_2}) + \dots + \mathcal{N}_{sc}(\rho_{A_1 | A_n}), \quad (42)$$

where  $\mathcal{N}_{sc}(|\psi\rangle_{A_1 | A_2 \dots A_n})$  is the *ons*-SCREN of  $|\psi\rangle_{A_1 A_2 \dots A_n}$  with respect to the bipartition between  $A_1$  and the other qudits, and  $\mathcal{N}_{sc}(\rho_{A_1 | A_s})$  is the two-SCREN of the two-qudit state  $\rho_{A_1 A_s}$  with  $s = 2, \dots, n$ .

*Proof.* For the one-SCREN of  $|\psi\rangle_{A_1 \dots A_n}$  with respect to the bipartition between  $A_1$  and the other qudits, the reduced density matrix  $\rho_{A_1}$  of  $|\psi\rangle_{A_1 \dots A_n}$  onto subsystem  $A_1$  is obtained as

$$\begin{aligned} \rho_{A_1} &= \text{tr}_{A_2 \dots A_n} |\psi\rangle_{A_1 A_2 \dots A_n} \langle \psi| \\ &= p \sum_{i,j=1}^{d-1} a_{1i} a_{1j}^* |i\rangle_{A_1} \langle j| + [p\Omega + (1-p)] |0\rangle_{A_1} \langle 0| \\ &\quad + \sqrt{p(1-p)} \left[ \sum_{i=1}^{d-1} a_{1i} |i\rangle_{A_1} \langle 0| + \sum_{j=1}^{d-1} a_{1j}^* |0\rangle_{A_1} \langle j| \right], \end{aligned} \quad (43)$$

where  $\Omega = \sum_{s=2}^n \sum_{i=1}^{d-1} |a_{si}|^2 = 1 - \sum_{j=1}^{d-1} |a_{1j}|^2$ .

From the definition of pure state negativity in Eq. (4) together with Eq. (43), we have the one-SCREN of  $|\psi\rangle_{A_1 A_2 \dots A_n}$  between  $A_1$  and the other qudits as

$$\begin{aligned} \mathcal{N}_{sc} \left( |\psi\rangle_{A_1 | A_2 \dots A_n} \right) &= \left( (\text{tr} \sqrt{\rho_{A_1}})^2 - 1 \right)^2 \\ &= 4p^2 (1 - \Omega) \Omega. \end{aligned} \quad (44)$$

For the two-SCREN's  $\mathcal{N}_{sc}(\rho_{A_1 A_s})$  with  $s = 2, \dots, n$  that appear the right-hand side of Eq. (42), we first consider the case when  $s = 2$ , where all the other cases are analogously following. The two-qudit reduced density matrix  $\rho_{A_1 A_2}$  of  $|\psi\rangle_{A_1 A_2 \dots A_n}$  is obtained as

$$\begin{aligned} \rho_{A_1 A_2} &= \text{tr}_{A_3 \dots A_n} |\psi\rangle_{A_1 A_2 \dots A_n} \langle \psi| \\ &= p \sum_{i,j=1}^{d-1} [a_{1i} a_{1j}^* |i0\rangle_{A_1 A_2} \langle j0| + a_{1i} a_{2j}^* |i0\rangle_{A_1 A_2} \langle 0j| + a_{2i} a_{1j}^* |0i\rangle_{A_1 A_2} \langle j0| + a_{2i} a_{2j}^* |0i\rangle_{A_1 A_2} \langle 0j|] \\ &\quad + (\Omega_2 + 1 - p) |00\rangle_{A_1 A_2} \langle 00| \\ &\quad + \sqrt{p(1-p)} \sum_{k=1}^{d-1} [(a_{1k} |k0\rangle + a_{2k} |0k\rangle)_{A_1 A_2} \langle 00| + a_{1k}^* |00\rangle_{A_1 A_2} (\langle k0| + a_{2k}^* \langle 0k|)], \end{aligned} \quad (45)$$

with  $\Omega_2 = 1 - \sum_{j=1}^{d-1} (|a_{1j}|^2 + |a_{2j}|^2)$ . We further note that, by considering two unnormalized states

$$\begin{aligned} |\tilde{x}\rangle_{A_1 A_2} &= \sqrt{p} \sum_{i=1}^{d-1} (a_{1i} |i0\rangle_{A_1 A_2} + a_{2i} |0i\rangle_{A_1 A_2}) \\ &\quad + \sqrt{1-p} |00\rangle_{A_1 A_2}, \\ |\tilde{y}\rangle &= \sqrt{\Omega_2} |00\rangle_{A_1 A_2}, \end{aligned} \quad (46)$$

$\rho_{A_1 A_2}$  in Eq. (45) can be represented as

$$\rho_{A_1 A_2} = |\tilde{x}\rangle_{A_1 A_2} \langle \tilde{x}| + |\tilde{y}\rangle_{A_1 A_2} \langle \tilde{y}|. \quad (47)$$

Now Proposition 1 implies that for any pure state decomposition

$$\rho_{A_1 A_2} = \sum_h |\tilde{\phi}_h\rangle_{A_1 A_2} \langle \tilde{\phi}_h|, \quad (48)$$

where  $|\tilde{\phi}_h\rangle_{A_1 A_2}$  is an unnormalized state in two-qudit subsystem  $A_1 A_2$ , there exists an  $r \times r$  unitary matrix ( $u_{hl}$ ) such that

$$|\tilde{\phi}_h\rangle_{A_1 A_2} = u_{h1} |\tilde{x}\rangle_{A_1 A_2} + u_{h2} |\tilde{y}\rangle_{A_1 A_2}, \quad (49)$$

for each  $h$ . For the normalized state  $|\phi_h\rangle_{A_1 A_2} = |\tilde{\phi}_h\rangle_{A_1 A_2} / \sqrt{p_h}$  with  $p_h = |\langle \tilde{\phi}_h | \tilde{\phi}_h \rangle|$ , the definition of pure state negativity in Eq. (4) leads us to the two-SCREN of

$|\phi_h\rangle_{A_1 A_2}$ ,

$$\begin{aligned} \mathcal{N}_{sc} \left( |\phi_h\rangle_{A_1 | A_2} \right) &= \frac{4}{p_h^2} p^2 |u_{h2}|^4 (1 - \Omega) (\Omega - \Omega_2) \\ &= \frac{4}{p_h^2} p^2 |u_{h2}|^4 (1 - \Omega) \sum_{i=1}^{d-1} |a_{2i}|^2 \end{aligned} \quad (50)$$

for each  $h$ .

From the definition of SCREN for mixed states in Eq. (30) together with Eq. (50), we have the two-SCREN of  $\rho_{A_1 A_2}$  as

$$\begin{aligned} \mathcal{N}_{sc}(\rho_{A_1 | A_2}) &= \left[ \min_{\{p_h, |\phi_h\rangle\}} \sum_h p_h \sqrt{\mathcal{N}_{sc} \left( |\phi_h\rangle_{A_1 | A_2} \right)} \right]^2 \\ &= \left[ \min_{\{p_h, |\phi_h\rangle\}} \sum_h 2p |u_{h2}|^2 \sqrt{(1 - \Omega) \sum_{i=1}^{d-1} |a_{2i}|^2} \right]^2 \\ &= 4p^2 (1 - \Omega) \sum_{i=1}^{d-1} |a_{2i}|^2. \end{aligned} \quad (51)$$

where the last equality is due to the choice of  $u_{h2}$  from the unitary matrix ( $u_{hl}$ ). Here we note that the minimum average of the square-root of SCREN in Eq. (51) does not depend on the choice of pure-state decomposition of  $\rho_{A_1 A_2}$ , so that we could circumvent the minimization problem therein.

By using an analogous method, we have the two-SCREN of two-qudit mixed state  $\rho_{A_1 A_s}$  as

$$\mathcal{N}_{sc}(\rho_{A_1|A_s}) = 4p^2(1 - \Omega) \sum_{i=1}^{d-1} |a_{si}|^2, \quad (52)$$

for each  $s = 2, \dots, n$ . Now Eqs. (44) and (52) leads us to

$$\begin{aligned} \mathcal{N}_{sc}(|\psi\rangle_{A_1|A_2\cdots A_n}) &= 4p^2(1 - \Omega) \Omega \\ &= 4p^2(1 - \Omega) \sum_{s=2}^n \sum_{i=1}^{d-1} |a_{si}|^2 \\ &= \sum_{s=2}^n \left[ 4p^2(1 - \Omega) \sum_{i=1}^{d-1} |a_{si}|^2 \right] \\ &= \sum_{s=2}^n \mathcal{N}_{sc}(\rho_{A_1|A_s}), \end{aligned} \quad (53)$$

which completes the proof.  $\square$

Theorem 2 implies that Inequality (17), the multi-qubit CKW inequality in terms of one, and two-SCREN, is still true and in fact saturated for the class of multi-qudit states in Eq. (36).

To check the validity of SCREN SM inequality in (31) for the class of states in Eq. (36), we first note that Inequality (31) can be decomposed as

$$\begin{aligned} \mathcal{N}_{sc}(|\psi\rangle_{A_1|A_2\cdots A_n}) &\geq \sum_{m=3}^{n-1} \sum_{\vec{j}_m} \mathcal{N}_{sc}(\rho_{A_1|A_{j_1^m} \cdots |A_{j_{m-1}^m}})^{m/2} \\ &\quad + \sum_{j=2}^n \mathcal{N}_{sc}(\rho_{A_1|A_j}), \end{aligned} \quad (54)$$

where the second summation of the first term on the right-hand side of the inequality runs over all the index vectors  $\vec{j}^m = (j_1^m, \dots, j_{m-1}^m)$  with  $3 \leq m \leq n-1$ .

By Theorem 2, the last term of the right-hand side and the left-hand side of of Inequality (54) are equal to each other for the class of states in Eq. (36). Thus this class of states are good candidates as possible counterexamples for stronger version of monogamy inequalities, that is, Inequality (31). Moreover, the validity of SCREN SM inequality for this class of states necessarily implies that Inequality (31) must be saturated, that is, the residual term

$$\sum_{m=3}^{n-1} \sum_{\vec{j}_m} \mathcal{N}_{sc}(\rho_{A_1|A_{j_1^m} \cdots |A_{j_{m-1}^m}})^{m/2} \quad (55)$$

in (54) is zero for the class of states in Eq. (36). The following theorem states the main result of this paper, the saturation of multi-qudit SM inequality for the class of states in in Eq. (36).

**Theorem 3.** *For the class of  $n$ -qudit states  $|\psi\rangle_{A_1 A_2 \cdots A_n}$  in Eq. (36) that is a superposition of a  $n$ -qudit generalized  $W$ -class state and the vacuum, the multi-qudit SM inequality of entanglement in terms of SCREN is saturated;*

$$\mathcal{N}_{sc}(|\psi\rangle_{A_1|A_2\cdots A_n}) = \sum_{m=2}^{n-1} \sum_{\vec{j}_m} \mathcal{N}_{sc}(\rho_{A_1|A_{j_1^m} \cdots |A_{j_{m-1}^m}})^{m/2}. \quad (56)$$

*Proof.* As mentioned, it is enough to show that the residual term in Eq. (55) is zero for the class of states in Eq. (36). In fact, we further show that

$$\mathcal{N}_{sc}(\rho_{A_1|A_{j_1^m} \cdots |A_{j_{m-1}^m}}) = 0 \quad (57)$$

for all the index vectors  $\vec{j}^m = (j_1^m, \dots, j_{m-1}^m)$  with  $3 \leq m \leq n-1$ , that is, all the  $m$ -SCREN for  $3 \leq m \leq n-1$  is zero for the  $m$ -qudit reduced density matrices  $\rho_{A_1 A_{j_1^m} \cdots A_{j_{m-1}^m}}$ .

We use the mathematical induction on  $m$ , and first consider the case when  $m = 3$ . For any index vector  $\vec{j}^3 = (j_1, j_2)$  with  $j_1, j_2 \in \{2, 3, \dots, n\}$  [30], the left-hand side of Eq. (57) becomes the three-SCREN of the three-qudit reduced density matrix  $\rho_{A_1 A_{j_1} A_{j_2}}$ ,

$$\begin{aligned} \mathcal{N}_{sc}(\rho_{A_1|A_{j_1}|A_{j_2}}) &= \left[ \min_{\{p_h, |\psi_h\rangle\}} \sum_h p_h \sqrt{\mathcal{N}_{sc}(|\psi_h\rangle_{A_1|A_{j_1}|A_{j_2}})} \right]^2, \end{aligned} \quad (58)$$

where the minimization is over all possible pure state decompositions of  $\rho_{A_1 A_{j_1} A_{j_2}}$ . Let us consider an optimal decomposition

$$\rho_{A_1 A_{j_1} A_{j_2}} = \sum_k q_k |\phi_k\rangle_{A_1|A_{j_1}|A_{j_2}} \langle \phi_k|, \quad (59)$$

realizing the three-SCREN of of  $\rho_{A_1 A_{j_1} A_{j_2}}$ ,

$$\begin{aligned} \mathcal{N}_{sc}(\rho_{A_1|A_{j_1}|A_{j_2}}) &= \left[ \sum_k q_k \sqrt{\mathcal{N}_{sc}(|\phi_k\rangle_{A_1|A_{j_1}|A_{j_2}})} \right]^2. \end{aligned} \quad (60)$$

Because  $\rho_{A_1 A_{j_1} A_{j_2}}$  is a three-qudit reduced density matrix of  $|\psi\rangle_{A_1 A_2 \cdots A_n}$  in Eq. (36), Lemma 1 implies that  $|\phi_k\rangle_{A_1 A_{j_1} A_{j_2}}$  in Eq. (59) is a superposition of a three-qudit generalized  $W$ -class state and vacuum for each  $k$ . Due to Theorem 2, we also note that CKW-type monogamy inequality in terms of SCREN is saturated by  $|\phi_k\rangle_{A_1 A_{j_1} A_{j_2}}$  in Eq. (59) for each  $k$ ;

$$\mathcal{N}_{sc}(|\phi_k\rangle_{A_1|A_{j_1}|A_{j_2}}) = \mathcal{N}_{sc}(\rho_{A_1|A_{j_1}}^k) + \mathcal{N}_{sc}(\rho_{A_1|A_{j_2}}^k), \quad (61)$$



where  $\rho_{A_1 A_{j_1}}^k$  and  $\rho_{A_1 A_{j_2}}^k$  are the reduced density matrices of  $|\phi_k\rangle_{A_1 A_{j_1} A_{j_2}}$  onto two-qudit subsystems  $A_1 A_{j_1}$  and  $A_1 A_{j_2}$  respectively.

From the definition of pure-state SCREN in Eq. (29) together with Eq. (61), we have

$$\begin{aligned} \mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} A_{j_2}}) &= \mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} A_{j_2}}) \\ &\quad - \mathcal{N}_{sc}(\rho_{A_1 A_{j_1}}^k) - \mathcal{N}_{sc}(\rho_{A_1 A_{j_2}}^k) \\ &= 0, \end{aligned} \quad (62)$$

for each three-qudit pure state  $|\phi_k\rangle_{A_1 A_{j_1} A_{j_2}}$  in Eq. (60), and thus we have

$$\mathcal{N}_{sc}(\rho_{A_1 A_{j_1} A_{j_2}}) = 0 \quad (63)$$

for any three-qudit reduced density matrix  $\rho_{A_1 A_{j_1} A_{j_2}}$  of  $|\psi\rangle_{A_1 A_2 \dots A_n}$  in Eq. (36).

Now we assume the induction hypothesis; for any  $(m-1)$ -qudit reduced density matrix  $\rho_{A_1 A_{j_1} A_{j_2} \dots A_{j_{m-2}}}$  of the state  $|\psi\rangle_{A_1 A_2 \dots A_n}$  in Eq. (36), we assume

$$\mathcal{N}_{sc}(\rho_{A_1 A_{j_1} A_{j_2} \dots A_{j_{m-2}}}) = 0. \quad (64)$$

For any index vector  $\vec{j} = (j_1, j_2, \dots, j_{m-1})$  with  $\{j_1, j_2, \dots, j_{m-1}\} \subseteq \{2, 3, \dots, n\}$  and the  $m$ -qudit reduced density matrix  $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ , we consider an optimal pure-state decomposition

$$\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}} = \sum_k q_k |\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle \phi_k| \quad (65)$$

realizing  $m$ -SCREN of  $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ , that is,

$$\begin{aligned} \mathcal{N}_{sc}(\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}) &= \left[ \sum_k q_k \sqrt{\mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}})} \right]^2. \end{aligned} \quad (66)$$

From the definition of pure-state SCREN in Eq. (29), the  $m$ -SCREN of each  $|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  in Eq. (66) is

$$\begin{aligned} \mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}) &= \mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}) \\ &\quad - \sum_{s=2}^{m-1} \sum_{\vec{i}^s} \mathcal{N}_{sc}(\rho_{A_1 A_{i_1} \dots A_{i_{s-1}}}^k)^{s/2}, \end{aligned} \quad (67)$$

where  $\rho_{A_1 A_{i_1} \dots A_{i_{s-1}}}^k$  is the reduced density matrix of  $|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  on  $s$ -qudit subsystems  $A_1 A_{i_1} \dots A_{i_{s-1}}$ , and the second summation is over all possible index vectors  $\vec{i}^s = (i_1, i_2, \dots, i_{s-1})$  with  $\{i_1, i_2, \dots, i_{s-1}\} \subseteq \{j_1, j_2, \dots, j_{m-1}\}$ . We further divide the last term of

the right-hand side of Eq. (67) into the summations of two-SCREN and the others;

$$\begin{aligned} \mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}) &= \mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}) \\ &\quad - \sum_{l=1}^{m-1} \mathcal{N}_{sc}(\rho_{A_1 A_{j_l}}^k) \\ &\quad - \sum_{s=3}^{m-1} \sum_{\vec{i}^s} \mathcal{N}_{sc}(\rho_{A_1 A_{i_1} \dots A_{i_{s-1}}}^k)^{s/2}. \end{aligned} \quad (68)$$

For each  $s = 3, \dots, m-1$ ,  $\rho_{A_1 A_{i_1} \dots A_{i_{s-1}}}^k$  in the last summation of Eq. (68) is a  $s$ -qudit reduced density matrix of the  $m$ -qudit state  $|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  where Lemma 1 implies that  $|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  in Eq. (65) is a superposition of a  $m$ -qudit W-class state and vacuum. Thus the induction hypothesis assures that the  $s$ -SCREN of  $\rho_{A_1 A_{i_1} \dots A_{i_{s-1}}}^k$  is zero;

$$\mathcal{N}_{sc}(\rho_{A_1 A_{i_1} \dots A_{i_{s-1}}}^k) = 0, \quad (69)$$

for each  $s = 3, \dots, m-1$  and the index vector  $\vec{i}^s = (i_1, i_2, \dots, i_{s-1})$ .

Furthermore, Theorem 2 implies that the CKW-type monogamy inequality in terms of one and two SCREN is saturated by  $|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ , that is,

$$\mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}) = \sum_{l=1}^{m-1} \mathcal{N}_{sc}(\rho_{A_1 A_{j_l}}^k), \quad (70)$$

for each  $k$ . From Eq. (68) together with Eqs. (69) and (70), we have

$$\mathcal{N}_{sc}(|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}) = 0 \quad (71)$$

for each  $|\phi_k\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  that arises in the decomposition of  $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  in Eq. (65). Thus Eqs. (66) and (71) lead us to

$$\mathcal{N}_{sc}(\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}) = 0, \quad (72)$$

for any the  $m$ -qudit reduced density matrix  $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$  of  $|\psi\rangle_{A_1 A_2 \dots A_n}$  with  $3 \leq m \leq n-1$ .  $\square$

## V. CONCLUSIONS

In this paper, we have proposed SCREN as a powerful candidate to characterize the strongly monogamous property of multi-qudit systems. We have provided a SM inequality of multi-party entanglement in terms of SCREN, and shown that the tangle-based SM inequality of multi-qubit systems can be rephrased by SCREN. We

have also shown that SCREN SM inequality is still true for the counterexamples of CKW inequality in higher-dimensional systems. We have further provided an analytical proof that SCREN SM inequality is saturated by a large class of multi-qudit states, a superposition of multi-qudit generalized W-class states and vacuums. Thus SCREN is a good alternative of tangle in characterizing strong monogamy of multi-party entanglement without any known counterexample even in higher-dimensional systems.

Noting the importance of the study on multi-party quantum entanglement, our result can provide a rich ref-

erence for future work on the study of entanglement in complex quantum systems.

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- for some orthonormal bases  $\{|e_0\rangle_A, |e_1\rangle_A\}$  and  $\{|f_0\rangle_B, |f_1\rangle_B\}$  of subsystems  $A$  and  $B$ , respectively.
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